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A black ring with a rotating 2-sphere

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Abstract: We present a solution of the vacuum Einstein's equations in five dimensions corresponding to a black ring with horizon topology $S^1 \times S^2$ and rotation in the azimuthal direction of the S^2. This solution has a regular horizon up to a conical singularity, which can be placed either inside the ring or at infinity. This singularity arises due to the fact that this black ring is not balanced. In the infinite radius limit we correctly reproduce the Kerr black string, and taking another limit we recover the Myers-Perry black hole with a single angular momentum.

Keywords: Black Holes in String Theory, Black Holes.

Contents

1. Introduction

The discovery of the black ring by [1] showed that the four-dimensional black hole theorems [2] do not have a simple extension to higher dimensions. Black rings have a horizon with topology $S^1 \times S^2$ while the Myers-Perry (MP) black hole [3] in five dimensions has a horizon with the topology of an S^3. On the other hand, in four dimensions, the black hole theorems state that the only allowed horizon topology is that of an S^2. Moreover, it was shown in [1] that both the black hole and the black ring can carry the same conserved charges, namely the mass and a single angular momentum, and hence there is no uniqueness theorem in five dimensions. Further studies on black rings showed that they can also carry nonconserved charges [4], which is a completely novel feature with respect to spherical black holes. The fact that black rings can have *dipole* charges gives rise to infinite continuous nonuniqueness at the classical level since the parameters describing these charges can be varied continuously without altering the conserved charges.

If string theory is the correct theory of quantum gravity, it should account for the microscopic states of both spherical black holes and black rings. Therefore, black rings constitute an interesting test ground for string theory. A step towards the understanding of black rings in string theory was taken in [5], following [6], where it was found that black rings were related to supertubes [7, 8], which are well-known objects within the string theory framework. The recent discovery of supersymmetric black rings [9] (see also [10 – 12]) and the generalization to non-BPS black rings with three charges and three dipoles [13], has provided further evidence of this relationship. Both the BPS and the non-BPS black rings with three charges and three dipoles have two angular momenta J_ψ and J_ϕ. In fact, [1] already conjectured the existence a neutral black ring with two independent angular momenta. Moreover, in [13] a further conjecture was made about the existence of a family of non-supersymmetric black rings with nine parameters $(M, J_\psi, J_\phi, Q_{1,2,3}, q_{1,2,3})$. Recently [14] conjectured the existence of an even larger family of black rings, which would depend on twenty-one parameters, up to duality transformations. Motivated by these conjectures, in this paper we present a neutral black ring with rotation in the azimuthal direction of the S^2, which from now on we call ϕ. Recall that the black ring of [1] has rotation on the plane of the ring along the direction of the S^1, which we call ψ. Our solution should be the $J_\psi \to 0$ limit of the more general yet to be found doubly spinning black ring. The solution we present in this paper provides new evidence in favour of the existence of this neutral black ring with two independent angular momenta.

Ref. [15] has independently constructed a black ring with rotation in the S^2. It is claimed that their solution reproduces, in appropriate limits, the Kerr black string and the MP black hole with a single angular momentum, as does ours and hence both solutions may be equivalent. However, this is not evident since the coordinates used in [15] are rather involved. Instead, we present the solution in essentially the same coordinates as in [4] so that the connections with the previously known black ring solutions are immediately apparent. Using these coordinates we study all its properties, including the different limits that this solution meets. Specifically, we find that our solution has a regular horizon except for the conical excess singularity inside the ring which prevents it from collapsing. We could also choose to place the conical defect at infinity, which would equally stabilize the configuration, but the resulting spacetime would not be asymptotically flat. The presence of this singularity can be understood intuitively; a black ring can be constructed by taking a black string, identifying its ends so as to form an S^1 and adding angular momentum in the direction of the S^1 in order to compensate the self-attraction and the tension of the string. In our case, since the angular momentum is on the plane orthogonal to the ring, it does not balance the configuration and a conical singularity inside the ring is required. Our solution has three independent parameters, namely R, which is roughly the radius of the ring, λ, which plays a similar role as the ν parameter in the static ring [16][1] and is related to the conical singularity, and a new parameter a with dimensions of length, which is associated to the angular momentum. Taking the limit $a \to 0$ of our solution, one obtains the static ring [16] in the form given in [4].

The rest of the paper is organized as follows. In section 2 we present the solution and compute the main physical properties. Also, we study the horizon geometry and derive the Smarr relation. In section 3 we study the different limits of our solution. Specifically, taking infinite radius limit we obtain the Kerr black string. Moreover, we also study the limit which connects our solution with the MP black hole with a single angular momentum. The conclusions are presented in section 4.

2. The black ring with a rotating S^2

In this section we present the black ring with rotation on the S^2. The solution has been obtained by educated guesswork, from limits it would be expected to reproduce. It looks

[1]Note that the C-metric coordinates used in [16, 1] are different from the ones used in [5, 17], which in turn are different from those in [4]. Throughout this paper we use the same coordinates as ref. [4].

rather similar to the other neutral black ring solutions, which involve simple polynomic functions of x and y. Here we study the physical properties of the solution and the horizon geometry.

2.1 The solution

The metric for the ϕ-spinning ring is

$$ds^2 = -\frac{H(\lambda, y, x)}{H(\lambda, x, y)} \left[dt - \frac{\lambda\, a\, y\, (1 - x^2)}{H(\lambda, y, x)}\, d\phi \right]^2 + \\ + \frac{R^2}{(x - y)^2}\, H(\lambda, x, y) \Bigg[- \frac{dy^2}{(1 - y^2) F(\lambda, y)} - \frac{(1 - y^2) F(\lambda, x)}{H(\lambda, x, y)}\, d\psi^2 + \\ + \frac{dx^2}{(1 - x^2) F(\lambda, x)} + \frac{(1 - x^2) F(\lambda, y)}{H(\lambda, y, x)}\, d\phi^2 \Bigg], \tag{2.1}$$

where

$$F(\lambda, \xi) = 1 + \lambda\, \xi + \left(\frac{a\, \xi}{R} \right)^2, \qquad H(\lambda, \xi_1, \xi_2) = 1 + \lambda\, \xi_1 + \left(\frac{a\, \xi_1\, \xi_2}{R} \right)^2. \tag{2.2}$$

The coordinates are the same as those used in [4], and their ranges are

$$1 < x < -1\,, \qquad -\infty < y < -1\,. \tag{2.3}$$

Recall that in this set of coordinates, ψ parametrizes an S^1 and (x, ϕ) an S^2, with $x \sim \cos\theta$, where θ is the polar angle, and ϕ the azimuthal angle. From (2.1) one sees that the black ring is rotating along the azimuthal direction of the S^2, while the previously known neutral black ring [1] rotates along the ψ direction of the S^1.

Throughout this paper we assume that

$$\frac{2a}{R} < \lambda < 1 + \frac{a^2}{R^2} \tag{2.4}$$

where the upper bound garantees that both g_{xx} and $g_{\phi\phi}$ are positive and hence there are no closed time-like curves [9, 10]. As we will show later on, the lower bound is required for there to be a horizon. Setting $a = 0$ in (2.1) gives the static ring first found in [16].

As in other black ring solutions, R corresponds roughly to the radius of the black ring, whereas a is a new parameter with dimensions of length which is related to the angular momentum, as will be seen from the expressions of the conserved charges. In fact, in section 3 it will become manifest that a is roughly equivalent to the rotation parameter of the four dimensional Kerr black hole.

In order to avoid a conical singularity at $x = -1$ and $y = -1$, we have to identify the periods of the angular coordinates ψ and ϕ according to

$$\Delta\phi = \Delta\psi = \frac{2\pi}{\sqrt{1 - \lambda + a^2/R^2}}\,. \tag{2.5}$$

Also, in order for the orbits of ∂_ϕ to close off smoothly at $x = 1$ we must require

$$\Delta\phi = \frac{2\pi}{\sqrt{1 + \lambda + a^2/R^2}}\,, \tag{2.6}$$

which cannot be satisfied at the same time as (2.5) for nonzero λ. For the asymptotically flat ring, eq. (2.5) holds and there is a conically singular disk at $x = 1$ supporting the ring. From now on we will stick to this case.

Asymptotic infinity is at $x, y \to -1$. Now, defining canonical angular variables,

$$\tilde{\psi} = \frac{2\pi}{\Delta\psi}\,\psi\,, \qquad \tilde{\phi} = \frac{2\pi}{\Delta\phi}\,\phi\,, \tag{2.7}$$

with $\Delta\psi$ and $\Delta\phi$ given in (2.5), and performing the coordinate transformation

$$\zeta = R\,\frac{\sqrt{2(-1-y)}}{x-y}\,, \qquad \eta = R\,\frac{\sqrt{2(1+x)}}{x-y}\,, \tag{2.8}$$

the metric (2.1) takes the manifestly flat form

$$ds^2 \sim -dt^2 + \zeta^2 d\tilde{\psi}^2 + d\zeta^2 + \eta^2 d\tilde{\phi}^2 + d\eta^2\,. \tag{2.9}$$

The ADM mass and angular momentum are given by

$$\begin{aligned} M &= \frac{3\pi R^2}{4G}\frac{\lambda}{1 - \lambda + a^2/R^2}\,, \\ J_\phi &= -\frac{\pi R^2}{G}\frac{\lambda a}{(1 - \lambda + a^2/R^2)^{3/2}}\,. \end{aligned} \tag{2.10}$$

Again, for $a = 0$ the angular momentum vanishes and the mass coincides with that of the static ring given in [4].

The Killing vector field ∂_t becomes spacelike at the least negative zero of $H(\lambda, y, x)$, y_e; this point defines the ergosurface. Similarly, the zeros of $F(\lambda, y)$

$$y_i = \frac{-\lambda - \sqrt{\lambda^2 - 4a^2/R^2}}{2a^2/R^2}\,, \qquad y_h = \frac{-\lambda + \sqrt{\lambda^2 - 4a^2/R^2}}{2a^2/R^2}\,, \tag{2.11}$$

determine the locations of the inner and outer horizons, respectively. In the limit $a \to 0$, we have $y_h \to -1/\lambda$, which is the location of the horizon for the static black ring, and $y_i \to -\infty$, for which the static ring has a curvature singularity. Both the ergosurface and the horizon have topology $S^1 \times S^2$, but we will come back to this point in the next subsection. Moreover, it can be checked that the Killing vector field ∂_ϕ remains spacelike throughout the ergoregion, which is the region $y_h < y < y_e$. As in the four-dimensional Kerr black hole, in the ϕ-spinning ring the ergoregion coincides with the horizon at the poles of the S^2 ($x = \pm 1$). Finally, from (2.11) it is clear that we must impose $\lambda > 2a/R$ so as to have a horizon.

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2.2 Horizon geometry

The Kretschmann invariant is finite at $y = y_i, y_h$, but blows up when $H(\lambda, x, y) = 0$. To find coordinates that are good on the horizon we proceed as in [9, 10] and define $\bar{y} = y - y_h$. Consider the following change of coordinates

$$dt = dv + \frac{A}{\bar{y}}\, d\bar{y}\,, \qquad d\phi = d\chi + \frac{B}{\bar{y}}\, d\bar{y}\,, \tag{2.12}$$

where A and B are determined by requiring that the solution in the coordinates $(v, x, \bar{y}, \psi, \chi)$ is analytic at $\bar{y} = 0$. When we expand near $\bar{y} = 0$, we find that $g_{v\bar{y}}$ and $g_{\chi\bar{y}}$ diverge as $1/\bar{y}$. Both divergences can be removed by imposing $A = -B\,\lambda\, R^2/(a\, y_h)$. This choice also removes the $1/\bar{y}^2$-divergence in $g_{\bar{y}\bar{y}}$. Then $g_{\bar{y}\bar{y}}$ diverges as $1/\bar{y}$ unless we impose

$$B^2 = \frac{y_h^2\; a^2/R^2}{(y_h^2 - 1)(\lambda^2 - 4a^2/R^2)}\,. \tag{2.13}$$

The metric is now analytic at $\bar{y} = 0$, and its form on a spatial cross-section of the horizon is rather simple:

$$ds^2_{\mathrm{H}} = \frac{R^2}{(x - y_h)^2}\left[(y_h^2 - 1)F(\lambda, x)\, d\psi^2 + \frac{H(\lambda, x, y_h)}{(1 - x^2)F(\lambda, x)}\; dx^2\right] + (1 - x^2)\frac{\lambda^2 R^2}{H(\lambda, x, y_h)} d\chi^2\,. \tag{2.14}$$

This expression shows that the horizon is regular, up to the conical singularity, and that its topology is indeed $S^1 \times S^2$, where the S^1 is parametrized by the angle ψ and the S^2 by (x, ϕ). Using (2.14) we can compute the horizon area

$$\mathcal{A}_{\mathrm{H}} = \frac{8\pi^2 R^3 \lambda}{(1 - \lambda + a^2/R^2)\sqrt{y_h^2 - 1}}\,. \tag{2.15}$$

For $a = 0$ this agrees with the $\nu = \lambda$ limit of the ψ-rotating black ring.

The Killing vector field

$$\partial_v - \frac{a\, y_h}{\lambda R^2}\, \partial_\chi \tag{2.16}$$

becomes null at $y = y_h$, and hence we can identify it as the null generator of the horizon. The angular velocity of the horizon and the temperature are given by

$$\Omega_\phi = -\frac{\sqrt{1 - \lambda + a^2/R^2}\,\left[\lambda - \sqrt{\lambda^2 - 4a^2/R^2}\right]}{2\lambda a}\,, \tag{2.17}$$

$$T_{\mathrm{BH}} = \frac{\sqrt{(\lambda^2 - 4a^2/R^2)(y_h^2 - 1)}}{4\pi R\lambda}\,. \tag{2.18}$$

Using these results, a straightforward calculation shows that the ϕ-spinning ring also satisfies a Smarr relation,

$$M = \frac{3}{2}\left[\frac{1}{4G}\mathcal{A}_{\mathrm{H}} T_{\mathrm{BH}} + \Omega_\phi J_\phi\right]. \tag{2.19}$$

This relationship holds in the case in which the solution is asymptotically flat, that is, the conical defect is inside the ring.

3. Limits

3.1 Infinite radius limit

In order to obtain the infinite radius limit we have to send $R \to \infty$ while keeping fixed

$$r = -\frac{R}{y}\,, \qquad \cos\theta = x\,, \qquad 2M = \lambda R\,, \qquad \eta = R\psi\,. \tag{3.1}$$

We then have

$$F(\lambda, x) \to 1\,, \qquad F(\lambda, y) \to \frac{\Delta}{r^2}\,, \qquad H(\lambda, x, y) \to \frac{\Sigma}{r^2}\,, \qquad H(\lambda, y, x) \to \frac{(\Sigma - 2Mr)}{r^2}\,, \tag{3.2}$$

with $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. In this limit the metric (2.1) becomes the Kerr black string,

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\left[dt + \frac{2Mr}{\Sigma - 2Mr} a\sin^2\theta\, d\phi\right]^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\Delta\sin^2\theta}{1 - \frac{2Mr}{\Sigma}} d\phi^2 + d\eta^2\,. \tag{3.3}$$

In this limit, the singularity encountered at $H(\lambda, x, y) = 0$ becomes the ring-like singularity of the Kerr spacetime.

3.2 Myers-Perry black hole limit

In order to recover the five-dimensional MP black hole with rotation in the ϕ direction from (2.1), we have to take a limit similar to the one described in [4]. We define new parameters

$$m = \frac{2(R^2 + a^2)}{1 + a^2/R^2 - \lambda}\,, \qquad \alpha^2 = \frac{4a^2}{1 + a^2/R^2 - \lambda}\,, \tag{3.4}$$

such that they remain finite as $\lambda \to 1 + a^2/R^2$, $R \to 0$ and $a \to 0$. Furthermore, we perform a change of coordinates $(x, y) \to (r, \theta)$

$$x = -1 + \frac{2R^2\cos^2\theta}{r^2 - (m - \alpha^2)\cos^2\theta}\,, \tag{3.5}$$

$$y = -1 - \frac{2R^2\sin^2\theta}{r^2 - (m - \alpha^2)\cos^2\theta}\,, \tag{3.6}$$

and rescale ψ and ϕ so that in this limit they have the canonical periodicity 2π,

$$(\tilde{\phi}, \tilde{\psi}) = \sqrt{\frac{2(R^2 + a^2)}{m}}\,(\phi, \psi)\,. \tag{3.7}$$

Then, we obtain the MP black hole with rotation in ϕ

$$ds^2 = -\left(1 - \frac{m}{\Sigma}\right)\left(dt - \frac{m\alpha\cos^2\theta}{\Sigma - m} d\tilde{\phi}\right)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\Delta\cos^2\theta}{1 - \frac{m}{\Sigma}} d\tilde{\phi}^2 + r^2\sin^2\theta\, d\tilde{\psi}^2\,, \tag{3.8}$$

where

$$\Delta = r^2 - m + \alpha^2\,, \qquad \Sigma = r^2 + \alpha^2\sin^2\theta\,. \tag{3.9}$$

4. Conclusions

In this paper we have presented a black ring solution with rotation on the plane orthogonal to the ring. Although our solution has a horizon with topology $S^1 \times S^2$, there is a conical singularity inside the ring that prevents it from collapsing. This solution should be an intermediate step towards a more general black ring with two independent angular momenta.

Using (2.1) as a seed solution and following the procedure outlined in [5, 13], one can construct the black ring with a rotating S^2 and carrying F1-P charges. In this charged black ring, which we do not present here, the conical singularity persists except in the extremal supersymmetric limit. In this limit, in which the angular momentum disappears, we find a configuration which can be interpreted as a continuous distribution of straight F1-P strings on a circle. We do not expect to be able to add more than two charges to the solution (2.1) without generating Dirac-Misner strings for the reasons discussed in [5]. Furthermore, it would be very interesting to construct the ϕ-spinning ring carrying dipole charges. Even then, we do not expect to be able to remove the conical singularity since the dipole charges typically increase the tension of the string [4]. Adding three charges to this more general solution should result in a charged ϕ-ring with dipole charges. We hope our results are of help towards a classification of the possible black holes of five-dimensional vacuum gravity. This is also a step in the construction of the seed solution that would generate the 9-parameter non-supersymmetric black rings conjectured in [13] (and, further, the larger family conjectured in [14]), that should describe thermal excitations of three-charge supertubes. We hope to tackle these problems in the future.

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